Math Instructional Framework

Full Name	Math III Unit 3 Lesson 2
Time Frame	
Unit Name	Logarithmic Functions as Inverses of Exponential Functions
Learning Task/Topics/ Themes	Task 2: How long Does It Take? Task 3: The Population of Exponentia Task 4: Modeling Natural Phenomena on Earth Culminating Task: Traveling to Exponentia
Standards **and** Elements	**MM3A2. Students will explore logarithmic functions as inverses of exponential functions.** c. Define logarithmic functions as inverses of exponential functions.
Lesson Essential Questions	How can you graph the inverse of an exponential function?
Activator	PROBLEM 2.Task 3: The Population of Exponentia (Problem 1 could be completed prior)
Work Session	**Inverse of Exponential Functions are Logarithmic Functions** **A** **Graph the inverse of exponential functions.** **B** **Graph logarithmic functions.** See Notes Below.
VOCABULARY	**Asymptote:** A line or curve that describes the end behavior of the graph. A graph never crosses a vertical asymptote but it may cross a horizontal or oblique asymptote. **Common logarithm:** A logarithm with a base of 10. A common logarithm is the power, a, such that $10^a = b$. The common logarithm of x is written $log\ x$. For example, $log\ 100 = 2$ because $10^2 = 100$. **Exponential functions:** A function of the form $y = a \cdot b^x$ where $a > 0$ and either $0 < b < 1$ or $b > 1$. **Logarithmic functions:** A function of the form $y = log_b x$, with $b \neq 1$ and b and x both positive. A logarithmic function is the inverse of an exponential function. The inverse of $y = b^x$ is $y = log_b x$. **Logarithm:** The logarithm base b of a number x, $log_b x$, is the power to which b must be raised to equal x. **Natural exponential:** Exponential expressions or functions with a base of e, i.e. $y = e^x$. **Natural logarithm:** A logarithm with a base of e. A natural logarithm is the power, a, such that $e^a = b$. The natural logarithm of x is written $ln\ x$. For example, $ln\ 8 = 2.0794415\ldots$ because $e^{2.0794415\ldots} = 8$.
Summarizing/Closing/Formative Assessment	PROBLEM 2.Task 3: The Population of Exponentia Additional Practice Exercise.

Inverse of Exponential Functions are Logarithmic Functions

A **Graph the inverse of exponential functions.**

B **Graph logarithmic functions.**

We have seen in Math 2 that the inverse function of a quadratic function is the square root function. In this section we examine inverse functions of exponential functions, called *logarithmic functions*.

A Graph the inverse of exponential functions.

What does the inverse graph of the graph of the exponential function $f(x) = 2^x$ look like? In the following example, we graph it using a t-table. Each point (x, y) on the graph of $f(x) = 2^x$ is plotted as (y, x) on its inverse graph.

Example 1. Graphing the inverse of an exponential function

(i) Plot the inverse graph of $f(x) = 2^x$.
(ii) Verify that a point (x, y) on f is plotted as (y, x) on its inverse graph.

Solution

(i)

Create a t-table to graph $\boldsymbol{y = 2^x}$.

Next Interchanges x and y values in your t-table, and graph these points as this is the inverse graph, $\boldsymbol{x = 2^y}$.

Graph the line $\boldsymbol{y = x}$.
Graphing the three above gives the result shown in Figure 1.

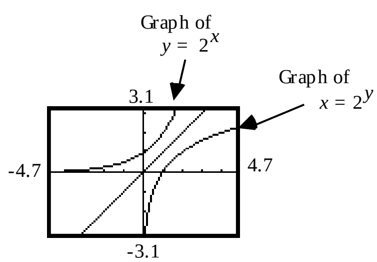


Figure 1.

(ii) Figure 2 shows the views when we *toggle* between the first two graphs (that is, trace to a point on one graph, then switch to the other graph).

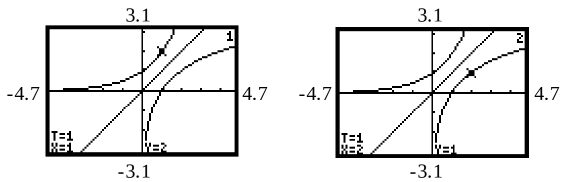


Figure 2. Graphs of $y = 2^x$, its inverse graph, and the line $y = x$

The point (1, 2) on the graph of $y = 2^x$ is reflected across the line $y = x$ as the point (2, 1) on its inverse graph, $x = 2^y$. Other points on the graph of $y = 2^x$ are similarly reflected to points on the inverse graph; in particular, the point (0, 1) is reflected as the point (1, 0).

Look at a table of values of the function $y = 2^x$.

x	-4	-3	-2	-1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

If we switch the rows, we get a table that displays some points on the inverse graph.

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
y	-4	-3	-2	-1	0	1	2	3	4

Plotting the points displayed in the tables and connecting them give the graphs of $y = 2^x$ and its inverse as shown in Figure 3. The x-axis is the horizontal asymptote of $y = 2^x$; the y-axis is the vertical asymptote of its inverse graph.

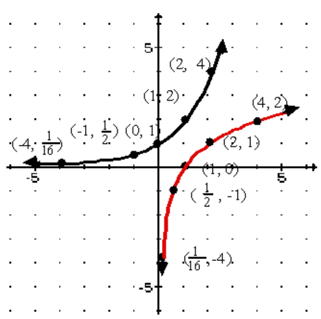


Figure 3. Some points on the graph of $y = 2^x$ reflected as points on its inverse graph

B Graph logarithmic functions.

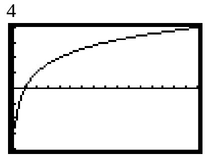
From the graphs and table of values, we see that $f(x) = 2^x$ is a one-to-one function. Therefore, its inverse graph also represents a function.

The inverse function of an exponential function is called a *logarithmic function*.

> *Definition* **Logarithmic Function**
>
> The inverse function of the exponential function $f(x) = b^x$ (where $b > 0$, $b \neq 1$) is called a **logarithmic function**, and is written
>
> $$f^{-1}(x) = \log_b x .$$
>
> We read $\log_b x$ as "log base b of x" or "log of x, base b."
> The domain of f^{-1} is $(0, \infty)$, and the range is $(-\infty, \infty)$.

As a result of the definition, the formula of the inverse function of $f(x) = 2^x$ is $f^{-1}(x) = \log_2 x$. The formula of the inverse function of $f(x) = 10^x$ is $f^{-1}(x) = \log x$. The subscript 10, denoting the base, is usually not written.

The four representations of the logarithmic function $f^{-1}(x) = \log_2 x$ are given below.

Rule	Equation	Table	Graph
Assign to the input value *x* an output whose value is log base 2 of *x*.	$y = \log_2 x$	x: $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8, 16 y: −2, -1, 0, 1, 2, 3, 4	

What exactly is a logarithm of a number? The point (0, 1) is on the graph of *f*, so we have $2^0 = 1$. Its reflected point (1, 0) is on the graph of f^{-1}, so we have $\log_2 1 = 0$. The point (1, 2) is on the graph of *f*, so we have $2^1 = 2$. Its reflected point (2, 1) is on the graph of f^{-1}, so we have $\log_2 2 = 1$. That is,

$2^0 = 1$ is equivalent to $\log_2 1 = 0$ and $2^1 = 2$ is equivalent to $\log_2 2 = 1$.

Continuing, we have

$2^2 = 4$ is equivalent to $\log_2 4 = 2$,
$2^3 = 8$ is equivalent to $\log_2 8 = 3$,
$2^4 = 16$ is equivalent to $\log_2 16 = 4$,

$2^{-1} = \frac{1}{2}$ is equivalent to $\log_2 \left(\frac{1}{2}\right) = {}^{-}1$.

Thus we see that the logarithm, base 2, of a number is the *exponent* whose power is that number.

exponent

$$2^3 = 8 \Leftrightarrow \log_2 8 = 3$$

power

In Math III, we'll evaluate logarithms. First, we graph logarithmic functions.

Technical Note: Change of Base Formula

Calculators have a built-in logarithmic function key, LOG, for the logarithm, base 10. To graph a logarithmic function having a base different than 10, we must use a **change of base formula**:

$$\log_b x = \frac{\log x}{\log b}.$$

For example, to graph $f(x) = \log_2 x$, we define *y* as log *x* ÷ log 2.

Let's first explore some graphs of logarithmic functions $f(x) = \log_b x$ where *b* > 1.

Example 2. Graphing $f(x) = \log_b x$, $b > 1$

Graph $f(x) = \log_b x$, for *b* = 1.5, 3, and 10. Describe any similarities and differences.

Solution

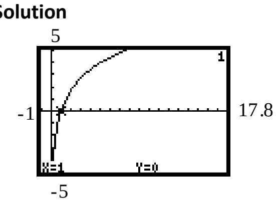


Figure 4. Graph of $f(x) = \log_{1.5} x$

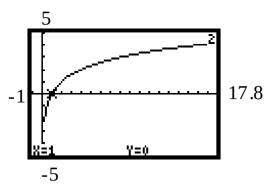


Figure 5. Graph of $f(x) = \log_3 x$

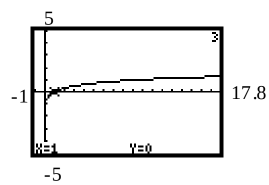


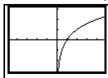
Figure 6. Graph of $f(x) = \log x$

By observing **Figures 4,5, and 6**, we see that for each graph, as *x* increases, so does *y*. The value $\log_{1.5} x$ increases faster than the value $\log_3 x$, and $\log x$ has the slowest growth of the three.

Since the three logarithmic functions are inverses of exponential functions *y* = 1.5^x, *y* = 3^x, and *y* = 10^x, the point (0, 1) on the exponential curves is reflected as the point (1, 0) on each logarithmic graph. The horizontal asymptote of the exponential curves is the *x*-axis. It is reflected as the vertical asymptote, *x* = 0 (*y*-axis), on the logarithmic curves.

The range of the exponential functions, $(0, \infty)$, becomes the domain of each logarithmic function. The domain of the exponential functions is the set of real numbers, thus the range of each logarithmic function is the set of real numbers. The curves are all continuous and smooth.

Below is a summary of the behavior of logarithmic functions whose base is greater than 1:



$f(x) = \log_b x$, $\boldsymbol{b > 1}$.

1. As x increases, so does y.
2. The y-axis is the vertical asymptote of the graph of f.
3. The graph passes through the point (1, 0).
4. The domain of f is $(0, \infty)$, and the range is $(-\infty, \infty)$.
5. The curve is continuous and smooth.

Let's look at some graphs of $f(x) = \log_b x$, where 0 < *b* < 1.

Example 3. Graphing $f(x) = \log_b x$, $0 < b < 1$

Graph $f(x) = \log_b x$, for *b* = 0.1, $\frac{1}{3}$, and 0.5. Describe any similarities and differences.

Solution

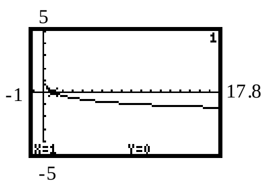


Figure 7. Graph of $f(x) = \log_{0.1} x$

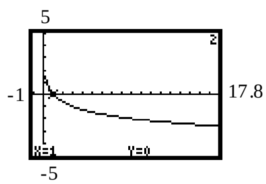


Figure 8. Graph of $f(x) = \log_{\frac{1}{3}} x$

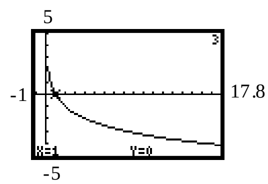
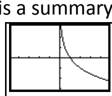


Figure 9. Graph of $f(x) = \log_{0.5} x$

By observing Figures 7, 8, and 9, we see that for each graph, as *x* increases, *y decreases*. The value $\log_{0.5} x$ decreases faster than the value $\log_{\frac{1}{3}} x$, and $\log_{0.1} x$ has the slowest decrease of the three.

The point (1, 0) is on each graph, and the *y*-axis is the asymptote. The domain of each function is $(0, \infty)$; the range is the set of real numbers. The three curves are all continuous and smooth.

Below is a summary of the behavior of logarithmic functions whose base is between 0 and 1:



$f(x) = \log_b x$, $\boldsymbol{0 < b < 1}$.

1. As x increases, y decreases.
2. The y-axis is the asymptote of the graph of f. As x decreases to near 0, y increases without bound, that is, y increases to $+\infty$.
3. The graph passes through the point (1, 0).
4. The domain of f is $(0, \infty)$, and the range is $(-\infty, \infty)$.
5. The curve is continuous and smooth.

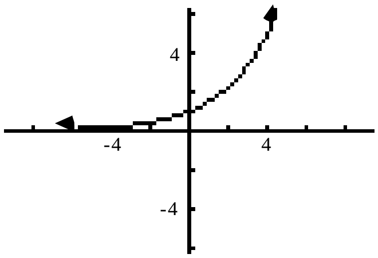
(Note that the only difference between the graphs of logarithmic functions $f(x) = \log_b x$ when *b* > 1 and $f(x) = \log_b x$ when 0 < *b* < 1 is that the former is an increasing function, and the latter is a decreasing function.)

Exercises Math III Unit 3

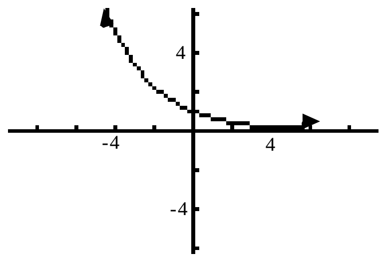
A **Graph the inverse of exponential functions.**

Exercises 1 - 2. Given the graph of *f*, sketch the graph of its inverse.

1.



2.



Comment [ART1]: y = (1.5)^x

Comment [ART2]: y = (1.5) ^ -x

Exercises 3 - 8. Plot the inverse graph of *f*.

3. $f(x) = 4^x$

5. $f(x) = 3^x$

5. $f(x) = 10^x$

6. $f(x) = 5^x$

7. $f(x) = 0.3^x$

8. $f(x) = 0.7^x$

Exercises 9 - 10. For the given table of a function, switch rows to get a table that displays some points on the inverse graph. Plot the points displayed in each of the two tables and connect the points with a smooth curve.

9.

x	-2	-1	0	1	2	3
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27

10.

x	-2	-1	0	1	2	3
y	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

Exercises Math III Unit 3

B Graph logarithmic functions.

Exercises 11 - 12. Determine the value of *u* and *w*.

11. **(a)** $3^0 = 1$ is equivalent to $\log_3 u = w$
(b) $3^1 = 1$ is equivalent to $\log_3 u = w$
(c) $3^2 = 9$ is equivalent to $\log_3 u = w$
(d) $3^3 = 27$ is equivalent to $\log_3 u = w$
(e) $3^4 = 81$ is equivalent to $\log_3 u = w$

12. **(a)** $10^0 = 1$ is equivalent to $\log u = w$
(b) $10^1 = 10$ is equivalent to $\log u = w$
(c) $10^2 = 100$ is equivalent to $\log u = w$
(d) $10^3 = 1{,}000$ is equivalent to $\log u = w$
(e) $10^4 = 10{,}000$ is equivalent to $\log u = w$

Exercises 13 - 16.

(i) **Write the equation of the inverse function of the given exponential function.**

(ii) **Verify your result visually by graphing both the exponential function and the inverse function, as well as the line *y* = *x*, on the same graph.**

13. $f(x) = 10^x$

14. $f(x) = 3^x$

15. $f(x) = 0.4^x$

16. $f(x) = 0.8^x$

Exercises 17 - 20. For the given three functions:

(i) **With a graphics calculator, graph each function, using the change of base formula.**

(ii) **Describe how the shape of the graph of $y = \log_b x$ changes as the value of *b* changes.**

(iii) **Determine the vertical asymptote for each function.**

(iv) **Determine the domain and range of each function.**

17. $y_1 = \log_{1.2} x$ $\quad y_2 = \log_4 x$ $\quad y_3 = \log x$

18. $y_1 = \log_{1.5} x$ $\quad y_2 = \log_3 x$ $\quad y_3 = \log_9 x$

19. $y_1 = \log_{0.1} x$ $\quad y_2 = \log_{\frac{2}{3}} x$ $\quad y_3 = \log_{0.9} x$

20. $y_1 = \log_{0.2} x$ $\quad y_2 = \log_{\frac{1}{2}} x$ $\quad y_3 = \log_{0.8} x$

Exercise 21 - 24. Match the logarithmic function with its graph.

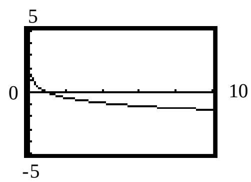
21. $f(x) = \log_5 x$

22. $g(x) = \log\ x$

23. $h(x) = \log_{0.5} x$

24. $k(x) = \log_{0.2} x$

(a)



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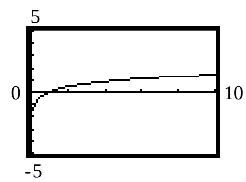

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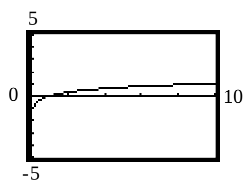

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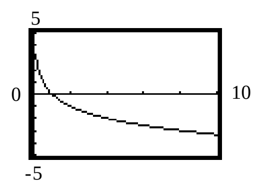


• Comprehensive and Continuation Exercises

25. Compare and contrast the *inverse function* of $y = x^2$ with the *inverse function* of $y = 2^x$.

Exercises 26 - 28. For the three given functions:

(i) **Graph each function.**

(ii) **Compare the graph of y_1 with the graphs of y_2 and y_3. Describe the changing positions of the graphs of y_2 and y_3 relative to the graph of y_1.**

(iii) **Determine the equations of their vertical asymptotes.**

(iv) **Determine their domain and range.**

26. $y_1 = \log_3 x$ $y_2 = \log_3(x + 2)$ $y_3 = \log_3(x - 5)$

27. $y_1 = \log x$ $y_2 = 5 + \log x$ $y_3 = {}^{-}3 + \log x$

28. $y_1 = \log_4 x$ $y_2 = {}^{-}\log_4 x$ $y_3 = \log_4({}^{-}x)$

Exercise 29 - 32. Match the logarithmic function with its calculator graph.

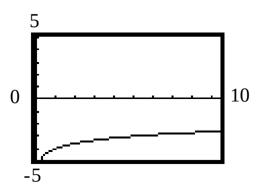
29. $f(x) = \log_5 x$

30. $f(x) = {}^{-}\log_5 x$

31. $f(x) = \log_5(x - 4)$

32. $f(x) = {}^{-}4 + \log_5 x$

(a)



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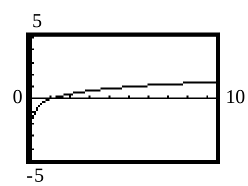

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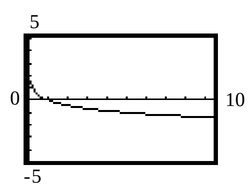

(d)