

Acquisition Lesson Planning Form
Key Standards addressed in this Lesson: MM2A3d,e
Time allotted for this Lesson: 4 Hours

Essential Question: LESSON 4 – FINITE ARITHMETIC SERIES AND RELATIONSHIP TO QUADRATIC FUNCTIONS
What are finite arithmetic series, how do you calculate their sums, and how are they related to quadratic functions?
Activating Strategies: (Learners Mentally Active)
<ul style="list-style-type: none">• Ask the students to add the first 100 natural numbers. Have them work in collaborative pairs using think-pair-share.
Acceleration/Previewing: (Key Vocabulary)
Vocabulary: Quadratic Inequalities, Series, Sequences
Teaching Strategies: (Collaborative Pairs; Distributed Guided Practice; Distributed Summarizing; Graphic Organizers)
Use the task and small groups with distributed guided practice. Discuss as a whole group the use of patterns to develop algebraic formulas. Then let the small groups work on sections of the task. Share with the entire group. Insert the graphic organizers where and when needed. Use the Graphic Organizers to discuss arithmetic sequences and series and partial sums of arithmetic series. Use the task to reinforce and teach about arithmetic series.
Distributed Guided Practice/Summarizing Prompts: (Prompts Designed to Initiate Periodic Practice or Summarizing)
What is the difference between a sequence and a series? What is the difference between an arithmetic sequence and a geometric sequence? How do you develop formulas for arithmetic and geometric sequences?
Extending/Refining Strategies:
Research geometric sequences and series. Present your findings to the class.
Summarizing Strategies: Learners Summarize & Answer Essential Question
Use parts of the task as a summarizer.

How do you find the n^{th} term of an arithmetic sequence?

Using given information, complete the chart.

$$\text{First term } a_1 = 3$$

$$\text{Common difference } d = 2$$

Position of term	a	expression
1 st	3	a_1
2 nd	3+2	a_1+d
3 rd	3+2+2	a_1+d+d
4 th	3+2+2+2	$a_1+d+d+d$
n^{th}		$a_1+(n-1)d$

Formula for the n^{th} term of an Arithmetic Sequence: $a_n = a_1 + (n-1)d$

How do you find the n^{th} term of an arithmetic sequence?

Using given information, complete the chart.

First term $a_1 = 3$

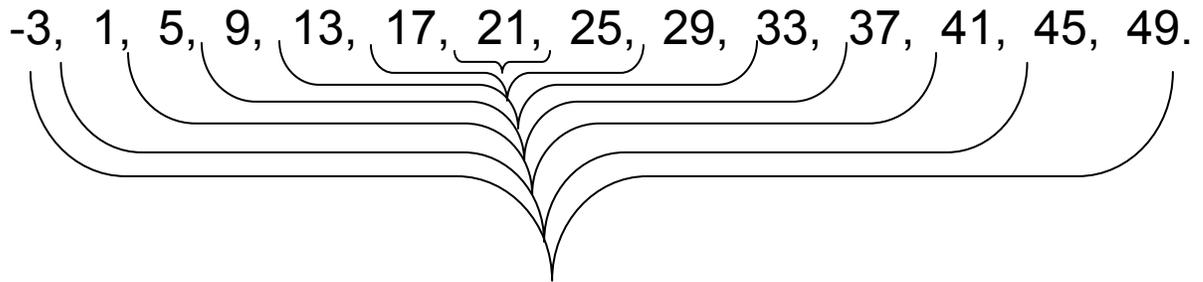
Common difference $d = 2$

Position of term	a	expression
1 st		
2 nd		
3 rd		
4 th		
n^{th}		

Formula for the n^{th} term of an Arithmetic Sequence:

How do you find the sum of an arithmetic series?

Look at the arithmetic series below.



Investigate the series and give the following information.

First term $a_1 =$ ____

Common difference $d =$ ____

Number of terms $n =$ ____

Last term $a_n =$ ____

The sum of each set of pairs marked above = ____

The sum of the series is ____

Would the following formula work? Explain your answer.

$$S = \frac{n}{2}(a_1 + a_n)$$

Suppose you did not know the last term. What could you do?

What are sequences of partial sums?

Look at the arithmetic series below.

$$-3, 1, 5, 9, 13, 17, 21, 25, 29, 33, \dots$$

We can find some partial sums of the series in the following manner.

$$S_1 = -3 \text{ the sum of 1 term}$$

$$S_2 = -3 + 1 = -2 \text{ the sum of 2 terms}$$

$$S_3 = -3 + 1 + 5 = 3 \text{ the sum of 3 terms}$$

$$S_4 = -3 + 1 + 5 + 9 = 12 \text{ the sum of 4 terms}$$

$$S_5 = -3 + 1 + 5 + 9 + 13 = 25 \text{ the sum of 5 terms}$$

$$S_6 = -3 + 1 + 5 + 9 + 13 + 17 = 42 \text{ the sum of 6 terms}$$

$$S_7 = -3 + 1 + 5 + 9 + 13 + 17 + 21 = 63 \text{ the sum of 7 terms}$$

$$S_8 = -3 + 1 + 5 + 9 + 13 + 17 + 21 + 25 = 88 \text{ the sum of 8 terms}$$

$$S_9 = -3 + 1 + 5 + 9 + 13 + 17 + 21 + 25 + 29 = 117 \text{ the sum of 9 terms}$$

$$S_{10} = -3 + 1 + 5 + 9 + 13 + 17 + 21 + 25 + 29 + 33 = 150 \text{ the sum of 10 terms}$$

If we write a series using these partial sums, then we have a **sequence**.

Sequence of partial sums is -3, -2, 3, 12, 25, 42, 63, 88, 117, 150, . . .

Consider the table of values made using the number of terms as the independent variable and the partial sums as the dependent variable. They form a quadratic equation.

x	1	2	3	4	5	6	7	8	9	10
y	-3	-2	3	12	25	42	63	88	117	150

Use your graphing calculator to do a quadratic regression using these points. The quadratic which fits this data is $y = 2x^2 - 5x$.

Write a sequence with at least 8 terms using the following series. Find the quadratic which represents this sequence.

$$-10, -5, 0, 5, 10, 15, 20, 25, \dots$$

Geometric Connections Learning Task

Scenario 1: A group of college freshmen attend a freshmen orientation session. Each is given a numbered “Hello” nametag. Students are told to shake hands with every other student there, and they do. How many handshakes are exchanged?

Scenario 2: An airline has several hub cities and flies daily non-stop flights between each pair of these cities. How many different non-stop routes are there?

Scenario 3: A research lab has several computers that share processing of important data. To insure against interruptions of communication, each computer is connected directly to each of the other computers. How many computer connections are there?

The questions asked at the end of the scenarios are really three specific versions of the same purely mathematical question: Given a set of objects, how many different pairs of objects can we form? One of the easiest ways to approach this problem involves thinking geometrically. The objects, whether they are college freshmen, cities, computers, etc., can be represented as points, and each pairing can be represented by drawing a line segment between the points. Such a representation is called a **vertex-edge graph**. You will explore vertex-edge graphs further in Mathematics III, but for now our focus is counting pairs of objects.

- Use a vertex-edge graph and an actual count of all possible edges between pairs of points to fill in the table below. When there are more than two vertices, it helps to arrange the points as if they are vertices of regular polygons as shown in the table.

Number of points/vertices	1	2	3	4	5
Example diagram of vertices					
Number of line segments/edges					

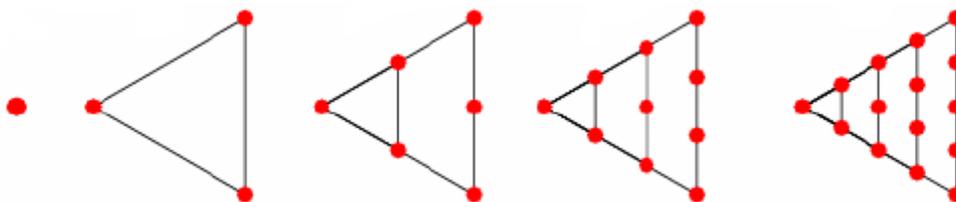
- Our goal is to find a formula that gives the number of pairs of objects as a function of n , the number of objects to be paired. Before we try to find the general formula, let's find the answers for a few more specific values.
 - Draw a set of 6 points and all possible edges among 5 of the points. How many edges do you have so far? How many additional edges do you draw to complete the diagram to include all possible edges between two points? What is the total number of edges?
 - Draw a set of 7 points and all possible edges among 6 of the points. How many edges do you have so far? How many additional edges do you draw to complete the diagram to include all possible edges between two points? What is the total number of edges?
 - We are trying to find a formula for the number of pairs of objects as a function of the number of objects. What is the domain of this function? Why does this domain allow us to think of the function as a sequence?

- d. Denote the number of pairs of n objects by p_n , so that the sequence is p_1, p_2, p_3, \dots . Remember that a recursive sequence is one that is defined by giving the value of at least one beginning term and then giving a recursive relation that states how to calculate the value of later terms based on the value(s) of one or more earlier terms. Generalize the line of reasoning from parts a and b to write a recursive definition of the sequence p_1, p_2, p_3, \dots . Which beginning term(s) do you need to specify explicitly? What is the recursive relation to use for later terms?
- e. Use your recursive definition to complete the table below and verify that it agrees with the previously found values.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p_n															

- f. Explain how the recursive definition of p_1, p_2, p_3, \dots also leads to expressing p_n as a sum of integers.

3. The numbers in the sequence p_2, p_3, p_4, \dots are known as the *triangular numbers*, so called because that number of dots can be arranged in a triangular pattern as shown below. (Image from [Weisstein, Eric W. "Triangular Number." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/TriangularNumber.html](http://mathworld.wolfram.com/TriangularNumber.html))



- a. The standard notation for the triangular numbers is T_1, T_2, T_3, \dots . Through exploration of patterns and/or geometric representations of these numbers, find a closed form formula for the n th triangular number, T_n .
- b. Explain the relationship between the sequences p_1, p_2, p_3, \dots and T_1, T_2, T_3, \dots .
- c. Use the relationship explained in part b to write a closed form formula for p_n .

4. Now we return to the scenarios.

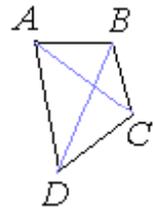
a. In Scenario 1, if there are 40 students at the orientation session, how many handshakes are exchanged?

b. In Scenario 2, if there are 190 airline routes between pairs of cities, how many cities are there in the group of hub cities?

c. In Scenario 3, if there are 45 computer connections, how many computers does the research lab use?

The formula you found in item 3, part d, and then applied in answering the questions of item 4, is well-known as the formula for counting **combinations of n objects taken two at a time**. We'll next explore another counting problem with geometric connections.

5. Remember that a **diagonal** of a polygon is a segment that connects non-adjacent vertices of a polygon. For example, in the quadrilateral $ABCD$, AC and BD are the diagonals.



a. Based on the definition above, complete the table below.

Number of sides/vertices in the polygon	3	4	5	6	7	8	9	n
Number of diagonals from each vertex								
Total number of diagonals in the polygon								

b. The last entry in the table provides a formula for the total number of diagonals in a polygon with n sides. Explain your reasoning for this formula.

6. Let d_0, d_1, d_2, \dots denote the sequence such that d_k is number of diagonals in a polygon with $k + 3$ sides.

a. Use the table form item 5, part a, to give the values for the first seven terms of the sequence: $d_0, d_1, d_2, d_3, d_4, d_5, d_6$.

b. We next seek a recursive definition of the sequence d_0, d_1, d_2, \dots . This recursive definition requires understanding another sequence known as the sequence of **first differences**. To aid our discussion of this other sequence, let f_1 denote the value added to d_0 to obtain d_1 , let f_2 denote the value added to d_1 to obtain d_2 , let f_3 denote the value added to d_2 to obtain d_3 , and so forth. Give the values for the first six terms of the sequence f_1, f_2, f_3, \dots .

c. The sequence f_1, f_2, f_3, \dots is an arithmetic sequence. Explain why.

d. The standard formula for the k^{th} term of an arithmetic sequence is $a + (k - 1)d$, where a is the first term and d is the common difference. What are a and d for the sequence f_1, f_2, f_3, \dots ? Write a formula for the k^{th} term of the sequence.

e. Write a recursive definition for the k^{th} term of the sequence d_0, d_1, d_2, \dots .

f. Explain why, for $k \geq 1$, we have the relationship that $d_k = f_1 + f_2 + f_3 + \dots + f_k$.

g. Use your formula for the number of diagonals of a polygon with n sides to write a formula for d_k .

h. Combine parts f and g to write a formula for sum of the first k terms of the sequence f_1, f_2, f_3, \dots .

7. So far in this task, you've worked with two formulas: one that gives the number of combinations of n objects taken 2 at a time and the other that counts the number of diagonals of a polygon with n sides. Since the first formula also gives the numbers of edges in the vertex-edge graph containing n vertices and all possible edges, there is a relationship between the two formulas. Explain the relationship geometrically and algebraically.

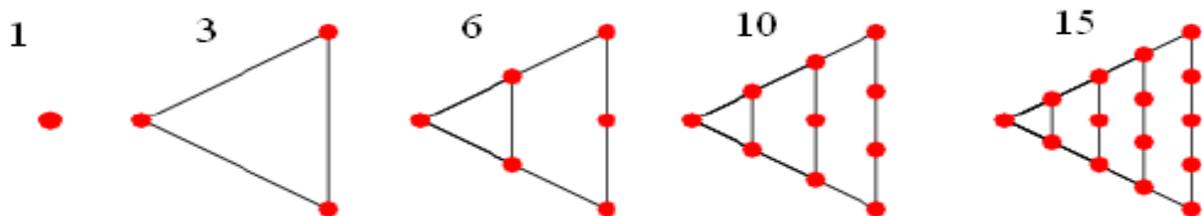
8. Each of the formulas studied so far provides a way to calculate certain sums of integers. One of the formulas sums the terms of the arithmetic sequence f_1, f_2, f_3, \dots . Look back at the sequence p_1, p_2, p_3, \dots from items 2 and 3.

a. What is the sequence of first differences for the sequence p_1, p_2, p_3, \dots ?

b. Is the sequence of first differences arithmetic? Explain.

c. What does this tell you about each term of the sequence p_1, p_2, p_3, \dots relative to arithmetic series?

In item 3, we recalled that summing up the terms of the sequence $1, 2, 3, \dots$ gives the *triangular numbers*, so called because that number of dots can be arranged in a triangular pattern as shown below. (Image from [Weisstein, Eric W. "Triangular Number." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/TriangularNumber.html](http://mathworld.wolfram.com/TriangularNumber.html)) The sequence $1, 2, 3, \dots$ starts with 1 and adds 1 each time. Next we consider what happens if we start with 1 and add 2 each time.



9. The sequence that starts with 1 and adds 2 each time is very familiar sequence.

a. List the first five terms of the sequence and give its common mathematical name.

b. Why is the sequence an arithmetic sequence?

c. Complete the table below for summing terms of this sequence.

Number of terms to be summed	Indicated sum of terms	Value of the sum
1	1	1
2	$1 + 3$	
3		
4		
5		
6		

d. Give a geometric interpretation for the sums in the table.

e. Fill in the last row of the table above when the number of terms to be summed is n , including a conjecture about the value for the last column. Use “. . .” in the indicated sum but include an expression, in terms of n , for the last term to be summed.

Our next goal is justification of the conjecture you just made in item 9, part e. However, rather than work on this specific problem, it just as easy to consider summing up any arithmetic sequence. The sum of the terms of a sequence is called a series so, as we proceed, we'll be exploring closed form formulas for **arithmetic series**.

10. Let a and d be real numbers and let a_1, a_2, a_3, \dots be the arithmetic sequence with first term a and common difference d . We'll use some standard notation for distinguishing between a sequence and the corresponding series which sums the terms of the sequence.

a. Fill in the table below with expressions in terms of a and d .

Term of the sequence	Express using a and d
a_1	
A_2	
A_3	
a_4	
a_5	
a_6	

b. Let s_1, s_2, s_3, \dots be the sequence defined by the following pattern:

$$\begin{aligned}
 s_1 &= a_1 \\
 s_2 &= a_1 + a_2 \\
 s_3 &= a_1 + a_2 + a_3 \\
 s_4 &= a_1 + a_2 + a_3 + a_4 \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 s_n &= a_1 + a_2 + a_3 + \dots + a_n
 \end{aligned}$$

Conjecture and prove a closed form formula for the arithmetic series s_n .

c. Use the sum of an arithmetic series formula to verify the conjecture for the sum of the arithmetic series in item 9.

11. Next we'll explore a different approach to a closed form formula for arithmetic series, one that is quite useful we are examining the type of series that occurs in a problem like the following. (This item is adapted from the "Common Differences" Sample Secondary Task related to K-12 Mathematics Benchmarks of The American Diploma Project, copyright 2007, Charles A. Dana Center at the University of Texas at Austin.)

Sam plays in the marching band and is participating in the fund raiser to finance the spring trip. He's taking orders for boxes of grapefruit to be delivered from a Florida grove just in time for the Thanksgiving and Christmas holiday season. With the help of his parents and grandparents, he is trying to win the prize for getting the most orders. The number of orders he got for each day of the last week are, respectively, 11, 15, 19, 23, 27, 31, 35. What is the total number of orders?

a. The total number of orders is the arithmetic series $11+15+19+23+27+31+35$. The numbers in the corresponding arithmetic sequence are plotted on a number line below.



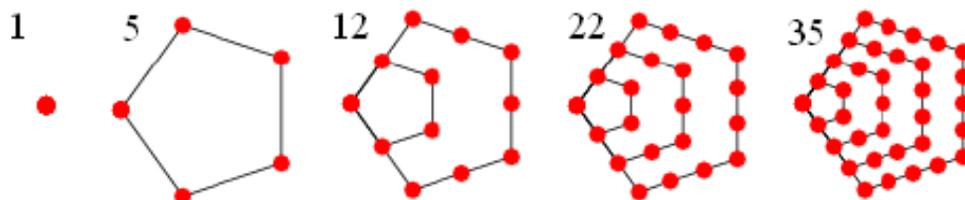
Find a shortcut for finding the sum of these seven terms. State your shortcut as a mathematical conjecture, and give a justification that your conjecture works.

b. Test your conjecture from part a on an arithmetic series with exactly 5 terms. Does it work for a series with exactly 9 terms? Does your conjecture apply to a series with exactly 6 terms?

c. Modify your conjecture, if necessary, so that it describes a general method for finding any finite sum of consecutive terms from an arithmetic sequence. Give a justification that your method will always work.

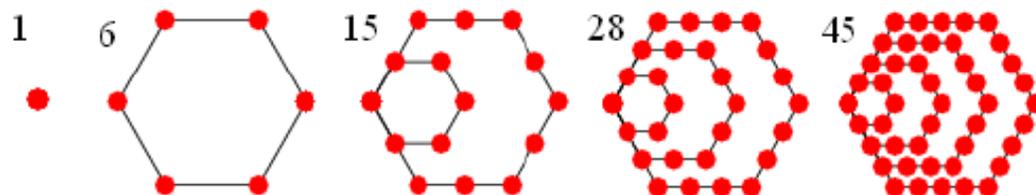
In addition to the well known triangular and square numbers, there are ***polygonal numbers*** corresponding to each type of polygon. The names relate to the figures that can be made from that number of dots. For example, the first five pentagonal, hexagonal, and heptagonal numbers are shown below.

Pentagonal numbers



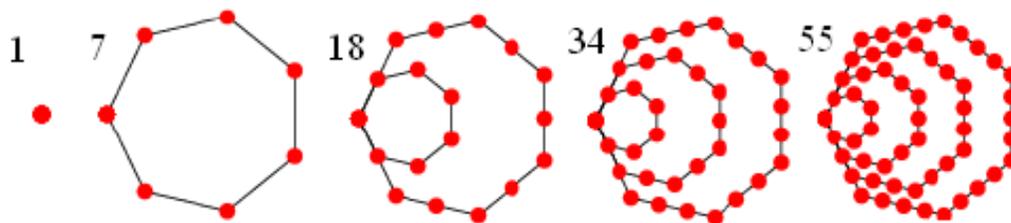
(Image from [Weisstein, Eric W. "Pentagonal Number." From MathWorld--A Wolfram Web Resource.](http://mathworld.wolfram.com/PentagonalNumber.html) <http://mathworld.wolfram.com/PentagonalNumber.html>)

Hexagonal numbers



(Image from [Weisstein, Eric W. "Hexagonal Number." From MathWorld--A Wolfram Web Resource.](http://mathworld.wolfram.com/HexagonalNumber.html) <http://mathworld.wolfram.com/HexagonalNumber.html>)

Heptagonal numbers



(Image from [Weisstein, Eric W. "Heptagonal Number." From MathWorld--A Wolfram Web Resource.](http://mathworld.wolfram.com/HeptagonalNumber.html) <http://mathworld.wolfram.com/HeptagonalNumber.html>)

12. As shown above, the number 1 is considered to be a polygonal number for each size polygon. The actual numbers in the sequence of each type of polygonal numbers are sums of an arithmetic series; for example, the n th triangular number is the sum of the arithmetic series $1+2+3+ \dots +n$. Use your knowledge of arithmetic sequences and series and of polygonal numbers complete the table below.

Polygonal number name	First five terms of the sequence of such numbers	Arithmetic sequence whose terms sum to form the numbers	Values of a and d	Formula for n^{th} polygonal number
Triangular				
Square				
Pentagonal				
Hexagonal				
Heptagonal				
Octagonal				

13. In 2001, the owner of two successful restaurants in south metropolitan Atlanta developed a long term expansion plan that led to the opening of five more restaurants in 2002. In 2003, eight new restaurants opened; in 2004, eleven new restaurants opened. Assume that the owner continued with this plan of increasing the number of new restaurants by three each year.

a. How many restaurants did the owner have in operation by the end of 2007?

b. If the owner continues with this plan, in what year will the 500th restaurant open?