Acquisition Lesson Planning Form
Key Standards addressed in this Lesson: MM2N1 a, b, c, d, and e
Time allotted for this Lesson: 5 hours

Essential Question: LESSON 2 – COMPLEX NUMBERS
What are complex numbers, how do you represent and operate using then?
Activating Strategies: (Learners Mentally Active)
• Historical story of i from "Imagining a New Number Learning Task," (This story ends before #1 on the task). • Students brainstorm the concepts from the previous day in small groups. Then share with the entire class.
Acceleration/Previewing: (Key Vocabulary)
Imaginary form, complex number, "i", standard form, pure imaginary number, complex conjugates, and complex number plane, absolute value of a complex number
Teaching Strategies: (Collaborative Pairs; Distributed Guided Practice; Distributed Summarizing; Graphic Organizers)
• Use the Imagining a New Number Learning Task. Insert the graphic organizers on pages as needed. Use small groups and collaborative pairs on the task along with distributed guided practice on teacher made practice sheets. • At the end of each day, groups should share with the class. Use a TOD as a final activity each day.
Distributed Guided Practice/Summarizing Prompts: (Prompts Designed to Initiate Periodic Practice or Summarizing)
• What exactly is the absolute value of any number? • How does the definition of absolute value apply to the complex plane? • Does the definition of absolute value as $\sqrt{a^2+b^2}$ work with real numbers? • How do you write a real number as a complex number?
Extending/Refining Strategies:
How would you solve $3x^2 - \sqrt{2}\,x + 2 = 0$ and $\sqrt{2}\,x^2 - 6x + \sqrt{8} = 0$
Summarizing Strategies: Learners Summarize & Answer Essential Question
Have the students answer the essential question and give examples. Selected problems from the graphic organizers might be used to summarize, perhaps as a ticket out the door. Use selected parts of the task as a summarizer each day.

GO # 1: Complex Numbers

Definition of an imaginary number: $i = \sqrt{-1}$

Squaring both sides: $i^2 = \left(\sqrt{-1}\right)^2$ which implies $i^2 = -1$

Hence:
$i^1 = i$
$i^2 = -1$
$i^3 = i \bullet i^2 = i \bullet -1 = -i$
$i^4 =$
$i^5 =$
$i^6 =$
$i^7 =$
$i^8 =$

What pattern do you see?

How could you find i^{19}?

What about i^{243}?

Remember simplifying square roots: $\sqrt{36}$ = ____ $-\sqrt{121}$ = _____

How would you simplify $\sqrt{-16}$? $\sqrt{-16} = \sqrt{16} \cdot \sqrt{-1} = 4i$

What about $\sqrt{-24}$? $\sqrt{-24} = \sqrt{4} \cdot \sqrt{6} \cdot \sqrt{-1} = 2\sqrt{6}\ i$ or $2i\sqrt{6}$

Now you try a few:

$\sqrt{-81}$ $\sqrt{-60}$ $-\sqrt{-12}$ $\sqrt{-\frac{9}{16}}$

$\sqrt{-4} - \sqrt{-36}$ $\sqrt{-8} + \sqrt{-18}$

GO # 2: Complex Number Form: a + bi

Write each of the following in complex number form (a + bi):

$6 + \sqrt{-18}$ = $\sqrt{8} - \sqrt{-9}$ =   3i =   12 =

Computing with Complex Numbers

To compute with radicals: Eliminate any powers of i greater than 1 and follow your rules for working with polynomials and radicals. Combine like terms. Use the rules for exponents with powers of i. Rationalize denominators.

Compute and simplify:

$\sqrt{25} - 2\sqrt{-36} + \sqrt{-4} - \sqrt{121}$ $(4i + 5) + (3 - 2i) - (7 - i)$

$3i(2 - 5i) + 6i(-3 + i)$ $14i^5 - 9i^4 + 7i^3 - 8i^2 + 12i - 3$

$-6i(3 - 2i)$ $(5 + 7i)(3 - 2i)$ $-i(2 + 4i)^2$

$(4 - 3i)^3$ $\frac{4 - 3i}{2i}$ $\frac{2 + 5i}{7 - 2i}$ $|\, 4 + 3i \,|$

Practice Problems on Solving Quadratics with Complex Solutions

1. $x^2 + 16 = 0$

2 $6x^2 + 42 = 0$

3. $2m^2 + 6x + 7 = 0$

4. $(x – 5)^2 = -16$

5. $2x^2 – 4x + 5 = 0$

6 $-7x^2 – 5x = 1$

7. $2x = 4 + 3x^2$

8. $4 – 3 (2x – 1)^2 = -20$

Imagining a New Number Learning Task

In other learning tasks of this unit, you encountered some quadratic equations for which the discriminates are negative numbers. For each of these equations, there is no real number solution because a solution requires that we find the square root of the discriminant and no real number can be the square root of a negative number. If there were a real number answer, the square of that number would have to be negative, but the square of every real number is greater than or equal to zero.

The problem of taking the square root of a negative number was ignored or dismissed as impossible by early mathematicians who encountered it. In his 1998 book, *The Story of* $\sqrt{-1}$*,* Paul J. Nahin describes the situation that most historians of mathematics acknowledge as the first recorded encounter with the square root of a negative number. Nahin quotes W. W. Beman, a professor of mathematics and mathematics historian from the University of Michigan, in a talk he gave at an 1897 meeting of the American Association for the Advancement of Science:

> We find the square root of a negative quantity appearing for the first time in the *Stetreometria* of Heron of Alexandria [*c.* 75 A.D.] . . . After having given a correct formula of the determination of the volume of a frustum of a pyramid with square base and applied it successfully to the case where the side of the lower base is 10, of the upper 2, and the edge 9, the author endeavors to solve the problem where the side of the lower base is 28, the upper 4, and the edge 15. Instead of the square root of 81 – 144 required by the formula, he takes the square root of 144 – 81 . . . , i.e., he replaces 1 by 1, and fails to observe that the problem as stated is impossible. Whether this mistake was due to Heron or to the ignorance of some copyist cannot be determined.

Nahin then observes that “so Heron missed being the earliest known scholar to have derived the square root of a negative number in a mathematical analysis of a physical problem. If Heron really did fudge his arithmetic then he paid dearly for it in lost fame.”

Nahin then reports on Diophantus of Alexandria, who most likely wrote his famous book, *Arithmetica,* about the year 250 A.D. Problem 22 of book 6 of the *Arithmetica* posed the question of finding the length of the legs of a right triangle with area 7 and perimeter 12, measured in appropriate units. Diophantus reduced this problem to that of solving the quadratic equation $336x^2 + 24 = 172x$. Diophantus knew a method of solving quadratic equations equivalent to the quadratic formula, but, quoting Nahin, “What he wrote was simply that the quadratic equation was not possible.”

1. Follow the steps below to see how Diophantus arrived at the equation $336x^2 + 24 = 172x$

    a. Let *a* and *b* denote the lengths of the legs of a right triangle with area 7 and perimeter 12. Explain why ab = 14 and $a + b + \sqrt{a^2 + b^2}$.= 12

b. Let *x* be a number so that a = $\frac{1}{x}$ and b = 14*x*. Based on the meaning of *a* and *b,* explain why there must be such a number *x*.

c. Replace *a* and *b* in the equation a + b + $\sqrt{a^2 + b^2}$.= 12 with the expressions in terms of *x*, and write an equivalent equation with the square root expression on the left side of the equation.

d. Square both sides of the final equation from part c and simplify to obtain Diophantus' equation: $336x^2 + 24 = 172x$

2. What happens when you use the quadratic formula to solve $336x^2 + 24 = 172x$?

According to Eugene W. Hellmich writing in Capsule 76 of *Historical Topics for the Mathematics Classroom, Thirty-first Yearbook of the National Council of Teachers of Mathematics*, 1969:

> The first clear statement of difficulty with the square root of a negative number was given in India by Mahavira (*c*. 850), who wrote: "As in the nature of things, a negative is not a square, it has no square root." Nicolas Chuquet (1484) and Luca Pacioli (1494) in Europe were among those who continued to reject imaginaries.

However, there was a break in the rejection of square roots of negative numbers in 1545 when Gerolamo (or Girolamo) Cardano, known in English as Jerome Cardan, published his important book about algebra, *Ars Magna* (Latin for “The Great Art”). Cardano posed the problem of dividing ten into two parts whose product is 40.

3. Note that, when Cardano stated his problem about dividing ten into two parts, he was using the concept of “divide” in the sense of dividing a line segment of length 10 into two parts of shorter length.

    a. Show that Cardano’s problem leads to the quadratic equation $x^2 - 10x + 40 = 0$.

    b. Find the solutions to this equation given by the quadratic formula even though they are not real numbers.

4. Rather than reject the solutions to $x^2 - 10x + 40 = 0$ as impossible, Cardano simplified them to obtain $5 + \sqrt{15}$ and $5 - \sqrt{15}$, stated that these solutions were “manifestly impossible”, but plunged ahead by saying “nevertheless, we will operate.” He “operated” by treating these expressions as numbers that follow standard rules of algebra and checked that they satisfied his original problem.

    a. Assuming that these numbers follow the usual rules of algebra, verify that their sum is 10.

    b. Assuming in addition that $\sqrt{-15} \bullet \sqrt{-15}$ = -15,verify that the product of the numbers is 40.

So, Cardano was the first to imagine that there might be some numbers in addition to the real numbers that we represent as directed lengths. However, Cardano did not pursue this idea. According to Hellmich in his mathematics history capsule, “Cardano concludes by saying that these quantities are “truly sophisticated” and that to continue working with them would be “as subtle as it would be useless.” Cardano did not see any reason to continue working with the numbers because he was unable to see any physical interpretation for numbers. However, other mathematicians saw that they gave useful algebraic results and continued the development of what today we call ***complex numbers***.

Cardano's *Ars Magna* drew much attention among mathematicians of his day, not because of his computations with the numbers 515and 515, but because it contained a formula, which came to be known as the Cardano formula, for solving any cubic equation.

5. State the standard form for a general cubic equation.
   a. Create some examples of cubic equations in standard form.

   b. Write formulas for cubic functions whose *x*-intercepts are the solutions of the cubic equations from part a.

   c. Use a graphing utility to graph the functions from part b. How many *x*-intercepts does each of the functions from part b have?

   d. How many real number solutions does each of the equations from part a have?

The Italian engineer Rafael Bombelli continued Cardano's work. In some cases, Cardano's formula gives roots of cubic equations expressed using the square root of a negative number. In his book, *Algebra,* published in 1572, Bombelli showed that the roots of the cubic equation x3 = 15x + 4 are 4, -2 + $\sqrt{3}$, and -2 - $\sqrt{3}$ but observed that Cardano's formula expresses the root of *x* = 4 as $\sqrt[3]{2+\sqrt{-121}} + \sqrt[3]{2-\sqrt{-121}}$. Quoting Nahin in *The Story of* $\sqrt{-1}$, "It was Bombelli's great insight to see that the weird expression that Cardano's formula gives for *x* is real, but expressed in a very unfamiliar manner." This realization led Bombelli to develop the theory of numbers complex numbers.

Today we recognize Bombelli's "great insight," but many mathematicians of his day (and some into the twentieth century) remained suspicious of these new numbers. Rene Descartes, the French mathematician who gave us the Cartesian coordinate system for plotting points, did not see a geometric interpretation for the square root of a negative number so in his book *La Geometrie* (1637) he called such a number "imaginary." This term stuck so that we still refer to the square root of a negative number as "imaginary." By the way, Descartes is also the one who coined the term "real" for the real numbers.

We now turn to the mathematics of these “imaginary” numbers.

In 1748, Leonard Euler, one of the greatest mathematicians of all times, started the use of the notation “*i*” to represent the square root of – 1, that is, *i* = $\sqrt{-1}$. Thus,

$$I^2 = \sqrt{-1} * \sqrt{-1} = -1,$$

since *i* represents the number whose square is – 1. This definition preserves the idea that the square of a square root returns us to the original number, but also shows that one of the basic rules for working with square roots of real numbers,

for any real numbers *a* and *b,* $\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}$

does not hold for square roots of negative numbers because, if that rule were applied we would not get –1 for i^2.

6. As we have seen, the number *i* is not a real number; it is a new number. We want to use it to expand from the real numbers to a larger system of numbers.

   a. What meaning could we give to i^2, i^3, i^4, i^5, ...?

   b. Find the square of each of the following: 2i, 7i, 10i, 25i.

   c. How could we use *i* to write an expression for each of the following: $\sqrt{-4}$, $\sqrt{-25}$, $\sqrt{-49}$?

   d. What meaning could we give ,-i. -2i, -3i, -4i, -5i, ...?

   e. Write an expression involving *i* for each of the following: $\sqrt{-9}$, $\sqrt{-16}$, $\sqrt{-81}$.

7. We are now ready to be explicit about imaginary numbers. An ***imaginary number*** is any number that can be written in the form *bi*, where *b* is an real number and I = $\sqrt{-1}$. Imaginary numbers are also sometimes called ***pure*** imaginary numbers.

   Write each of the following imaginary numbers in the standard form *bi*:

   $\sqrt{-\frac{1}{36}}$, $\sqrt{-11}$, $\sqrt{\frac{-5}{64}}$, $-\sqrt{-7}$, $-\sqrt{-18}$

8. Any number that can be written in the form $a + bi$, where a and b are real numbers, is called a ***complex number***. We refer to the form $a + bi$ as the ***standard form*** of a complex number and call a the ***real part*** and b the ***imaginary part*** .

Write each of the following as a complex number in standard form and state its real part and its imaginary part. 6 - $\sqrt{-1}$, -12 + $\sqrt{-100}$, 31 - $\sqrt{-20}$

As defined above, the set of complex numbers includes all of the real numbers (when the imaginary part is 0), all of the imaginary numbers (when the real part is 0), and lots of other numbers that have nonzero real and imaginary parts.

We say that two complex numbers are ***equal*** if their real parts are equal and their imaginary parts are equal, that is,

if $a+ bi$ and $c+ di$ are complex numbers,

then $a+ bi = c+ di$ if and only if $a = c$ and $b = d$.

In order to define operations on the set of complex numbers in a way that is consistent with the established operations for real numbers, we define addition and subtraction by combining the real and imaginary parts separately:

` if $a + bi$ and $c + di$ are complex numbers,

then $(a+ bi) + (c+ di) = (a+ c) + (b+ d)i$

$(a+ bi) - (c+ di) = (a- c) + (b- d)i$

9. Apply the above definitions to perform the indicated operations and write the answers in standard form.

a. (3 + 5i) + (2 – 6i)

b. (5 – 4i) – (-3 + 5i)

c. 13i – (-3 + 5i)

d. -7 – 9i - $\sqrt{-2}$

e. (5.4 + 8.3i) + (-3.7 + 4.6i)

f. $\left(\frac{3}{7}-\frac{5}{7}i\right)-\left(\frac{4}{7}+\frac{4}{7}i\right)$

To multiply complex numbers, we use the standard form of complex numbers, multiply the expressions as if the symbol *i* were an unknown constant, and then use the fact that $i^2 = -1$ to continue simplifying and write the answer as a complex number in standard form.

10. Perform each of the following indicated multiplications, and write your answer as a complex number in standard form. Note writing *i* in front of an imaginary part expressed as a root is standard practice to make the expression easier to read.

a. $6i(2 – i)$

b. $(2 - i)(3 + 4i)$

c. $\frac{2}{7}(9 – 6i)$

d. $(4 - i\sqrt{13})(2 + i\sqrt{13})$

e. $i\sqrt{5} \cdot i\sqrt{5}$

f. $\sqrt{-6} \cdot \sqrt{-12}$

11. For each of the following, use substitution to determine whether the complex number is a solution to the given quadratic equation.

a. Is $-2+3i$ a solution of $x^2 + 4x + 13 = 0$? Is $-2-3i$ a solution of this same equation?

b. Is $2+i$ a solution of $x^2 – 3x + 3 = 0$? Is $2i$ a solution of this same equation?

12. Find each of the following products.

a. $(1 + i)(1 – i)$

b. $(5 – 2i)(5 + 2i)$

c. $(\sqrt{7} + 8i)(\sqrt{7} - 8i)$

d. $\left(-\frac{1}{2}-\frac{2}{3}i\right)\left(-\frac{1}{2}+\frac{2}{3}i\right)$

e. $(4 + i\sqrt{6})(4 - i\sqrt{6})$

13. The complex numbers *a+ bi* and *a- bi* are called ***complex conjugates***. Each number is considered to be the complex conjugate of the other.

a. Review the results of your multiplications in item 12, and make a general statement that applies to the product of any complex conjugates.

b. Make a specific statement about the solutions to a quadratic equation in standard form when the discriminant is a negative number.

So far in our work with operations on complex numbers, we have discussed addition, subtraction, and multiplication and have seen that, when we start with two complex numbers in standard form and apply one of these operations, the result is a complex number that can be written in standard.

14. Now we come to division of complex numbers. Consider the following quotient: $\frac{-2+3i}{3-4i}$.

This expression is not written as a complex number in standard form; in fact, it is not even clear that it can be written in standard form. The concept of complex conjugates is the key to carrying out the computation to obtain a complex number in standard form.

a. Find the complex conjugate of 3-4*i*, which is the denominator of $\frac{-2+3i}{3-4i}$

.b. Multiply the numerator and denominator of $\frac{-2+3i}{3-4i}$ by the complex conjugate from part a.

c. Explain why $\frac{-2+3i}{3-4i} = \frac{-2+3i}{3-4i} \bullet \frac{3+4i}{3+4i}$ and then use your answers from part b obtain $-\frac{18}{25}+\frac{1}{25}i$ as the complex number in standard form equal to the original quotient.

d. For real numbers, if we multiply the quotient by the divisor we obtain the dividend. Does this relationship hold for the calculation above?

15. Use the same steps as in item 14 to simplify each of the following quotients and give the answer as a complex number in standard form.

a. $(5 + 6i) \div (2 + i)$

b. $\dfrac{14+2i}{2i}$

c. $\dfrac{3\sqrt{2}}{7-i\sqrt{2}}$

d. $\dfrac{2+\sqrt{-8}}{-5+\sqrt{-18}}$

Item 15 completes our discussion of basic numerical operations with complex numbers. In our discussion of the early history of the development of complex numbers, we noted that Cardano did not continue work on complex numbers because he could not envision any geometric interpretation for them. It was over two hundred years until, in 1797, the Norwegian surveyor Caspar Wessel presented his ideas for a very simple geometric model of the complex numbers. For the remainder of the task you will investigate the modern geometric representation of complex numbers, based on Wessel's work and that of Wallis, Argand, Gauss, and other mathematicians, some of whom developed the same interpretation as Wessel independently.

In representing the complex numbers geometrically, we begin with a number line to represent the pure imaginary numbers and then place this number line perpendicular to a number line for the real numbers. The number $0i = 0$ is both imaginary and real, so it should be on both number lines. Therefore, the two number lines are drawn perpendicular and intersecting at 0, just as we do in our standard coordinate system. Geometrically, we represent the complex number $a+bi$ by the point (a, b) in the coordinate system. When we use this representation, we refer to a complex number as a point in the ***complex number plane.*** Note that this is a very different interpretation for points in a plane that the one we use for graphing functions whose domain and range are subsets of the real numbers as we did in considering cubic functions in item 5.

16. Use a complex number plane to graph and label each of the following complex numbers:

$2+3i$, $-8-6i$, $-5i$, -2, $7-3i$, $6i$, $-3+4i$, 3.5

17. Geometrically, what is the meaning of the absolute value of a real number? We extend this idea and define the ***absolute value of the complex number* |*a*+ *bi*|** to be the distance in the complex number plane from the number to zero. We use the same absolute value symbol as we did with real numbers so that *abi* represents the absolute value of the complex number *abi*.

a. Find the absolute value of each of the complex numbers plotted in item 16.

b. Verify your calculations from part a geometrically.

c. Write a formula in terms of *a* and *b* for calculating |*a*+ *bi*|.

d. What is the relationship between the absolute value of a complex number and the absolute value of its conjugate? Explain.

e. What is the relationship between the absolute value of a complex number and the product of that number with its conjugate? Explain.

18. The relationship you described in part e of item 17 is a specific case of a more general relationship involving absolute value and products of complex numbers. Find each of the following products of complex numbers from item 16 and compare the absolute value of each product to the absolute values of the factors in the product. (Note that you found the length of each of the factors below in item 17, part a.) Finally, make and prove a conjecture about the absolute value of the product of two complex numbers.

a. 2 + 3i, 7 – 3i

b. -3 + 4i, 5i

c. -2, -8 – 6i

d. 6i, 3.5

19. By definition, $i^1 = I$ and $i^2 = -1$.

a. Find i^3, i^4, i^5, . . . , i^{12}. What pattern do you observe?

b. How do the results of your calculations of powers of *i* relate to item 18?

c. Devise a way to find any positive integer power of *i*, and use it to find the following powers of *i*: i^{26}, i^{55}, i^{136}, i^{373}.

20. We conclude this task with an exploration of the geometry of multiplying a complex number by *i* .

a. Graph i, i^2, i^3, i^4 in the complex number plane. How does the point move each time it is multiplied by *i* ?

b. Let $z = 5 + 12i$. What is iz?

c. Plot each of the complex numbers in part b and draw the line segment connecting each point to the origin.

d. Explain why multiplying by *i* does not change the absolute value of a complex number.

e. What geometric effect does multiplying by *i* seem to have on a complex number? Verify your answer for the number *z* from part b.